

ON THE ESSENTIAL SPECTRUM OF TWO-DIMENSIONAL PAULI OPERATORS WITH REPULSIVE POTENTIALS

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ABSTRACT. We investigate the spectrum of the two-dimensional Pauli operator, describing a spin- $\frac{1}{2}$ particle in a magnetic field B , with a negative scalar potential V such that $|V|$ grows at infinity. In particular, we obtain criteria for discrete and dense pure-point spectrum.

1. INTRODUCTION

For modeling the kinetic energy of a non-relativistic spin- $\frac{1}{2}$ particle in the plane, moving under a magnetic field B in the perpendicular direction to the plane, one uses the two-dimensional Pauli operator

$$H_{\mathbf{A}} := [\boldsymbol{\sigma} \cdot (-i\nabla - \mathbf{A})]^2 = (-i\nabla - \mathbf{A})^2 - \sigma_3 B \quad \text{on } L^2(\mathbb{R}^2, \mathbb{C}^2),$$

where \mathbf{A} is a vector potential associated to B , i.e. $B = \text{curl } \mathbf{A} := \partial_1 A_2 - \partial_2 A_1$. Here, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ and σ_3 are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To study the behaviour of such spin- $\frac{1}{2}$ particles (e.g. electrons) in presence of an additional electric potential V , we investigate the spectrum of the operator

$$H := H_{\mathbf{A}} + V = [\boldsymbol{\sigma} \cdot (-i\nabla - \mathbf{A})]^2 + V \quad \text{on } L^2(\mathbb{R}^2, \mathbb{C}^2).$$

If V is non-negative or decays at infinity (e.g. potentials with Coulomb singularities), spectral properties of the magnetic Schrödinger operator $(-i\nabla - \mathbf{A})^2 + V$, as well as of the Pauli operator $H_{\mathbf{A}} + V$ (in dimension $d = 2$ or 3) have been widely studied over the last decades (see, e.g. [2], [4] or [5] for a latest overview). In this article, instead, we want to point out some interesting features of the spectrum of H for potentials V tending to $-\infty$ as $|\mathbf{x}| \rightarrow \infty$. Since such scalar potentials result in an operator H , unbounded from below, it is necessary to discuss questions related to the self-adjointness of H . We emphasize that, since we also consider unbounded magnetic fields B , the self-adjointness of H cannot simply be reduced to the one of the magnetic Schrödinger operator.

One motivation for the following considerations is an observation made in [10] for the two-dimensional massless magnetic Dirac operator coupled to an electric potential V ; There, an accumulation process of spectral points has been observed, governed by the ratio $|V^2/B|$ at infinity. This phenomenon can be ascribed to the non-confining effect of V in the case of the Dirac operator. Regarding the Pauli

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operator, the influence of an additional scalar potential V on the spectrum $\sigma(H)$ depends crucially on the sign of V . For simplicity, we outline this dependence in the case of a constant magnetic field $B(x) = B_0$: a positive potential V growing at infinity always leads to discrete spectrum of the operator H , independently of the field strength B_0 (see e.g. [9]). Such trapping potentials only enhance a localization effect caused by B that generates eigenvalues and spectral gaps (proportional to B_0). If we instead consider negative potentials V , the situation is quite different since the particle lowers its energy by moving in regions where V is small. A scalar potential V converging to $-\infty$ as $|\mathbf{x}| \rightarrow \infty$ has therefore a delocalizing effect, i.e. the particle tends to escape any compact region of the plane. Our results show that such negative potentials V (describing for example constant radial fields) counteract the localizing effect of “hard” magnetic fields B , as they close spectral gaps induced by B :

- If V converges to $-\infty$, but remains small compared to B , the spectrum $\sigma(H)$ is discrete, i.e. it consists only of eigenvalues of finite multiplicity.
- If V is comparable to B , more precisely $|V| \approx 2B$ at infinity, points in the essential spectrum occur.
- If V overtakes B , more explicitly $|V/B| \rightarrow \infty$ as $|\mathbf{x}| \rightarrow \infty$ (at least along a path), the spectrum $\sigma(H)$ covers the whole real line.

One may compare the third claim with the result in [11] on $H_{\mathbf{A}}$ for decaying magnetic fields. The precise statements of the claims above are contained in Theorems 1–4 of Sect. 3. We remark that the case $|V/B| \rightarrow \infty$ as $|\mathbf{x}| \rightarrow \infty$ is treated by Theorems 3 and 4. Unlike Theorem 4, which is only valid for constant magnetic fields, Theorem 3 covers also non-constant fields B , but requires stronger constraints on the growth of V . Thus, the important case $B = B_0$ is addressed by two theorems. The ideas of the proofs of Theorem 1–3 originate from those used to prove the results in [10]. However, since we work with a second-order operator, the proofs are technically more laborious. Theorem 4 is based on a further construction of a Weyl sequence, obtained by treating V locally as a potential of a constant electric field. This is a refined ansatz compared to the method used for the proof of Theorem 3.

The organization of this article is as follows: In the next section some known facts about the Pauli operator are recapitulated. We present our precise results in Section 3, provided with some remarks and important applications. In Section 4 we give the proof of Theorem 1. The proofs of Theorems 2 and 3 are contained in Section 5, while the proof of Theorem 4 can be found in the last section. In the appendix, attached to the main text, we give a proof of the essential self-adjointness of the Pauli operator.

2. BASIC PROPERTIES OF THE PAULI OPERATOR

In this section we point out some basic facts about the Pauli operator and the massless Dirac operator $D_{\mathbf{A}}$, whose square equals $H_{\mathbf{A}}$. For a vector potential $\mathbf{A} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ generating the field $B = \text{curl } \mathbf{A} \in C(\mathbb{R}^2, \mathbb{R})$, the Hamiltonian $D_{\mathbf{A}}$ is defined as the closure of the operator

$$(1) \quad \sigma \cdot (-i \nabla - \mathbf{A}) = \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} \quad \text{on } C_0^\infty(\mathbb{R}^2, \mathbb{C}^2),$$

which is essentially self-adjoint on the given core (see [3]). In particular, d and d^* can be seen as closed operators, i.e. we use the notation

$$(2) \quad d = \overline{-i\partial_1 - A_1 + i(-i\partial_2 - A_2)} \upharpoonright_{C_0^\infty(\mathbb{R}^2, \mathbb{C})}$$

and analogously for d^* . One observes that d, d^* satisfy the commutation relation

$$(3) \quad [d, d^*]\varphi := (dd^* - d^*d)\varphi = 2B\varphi \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}).$$

We can write

$$H_{\mathbf{A}} = D_{\mathbf{A}}^2 = \begin{pmatrix} d^*d & 0 \\ 0 & dd^* \end{pmatrix} \quad \text{on } C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$$

and consider $H_{\mathbf{A}}$ as a self-adjoint operator on $\{\psi \in \mathcal{D}(D_{\mathbf{A}}) \mid D_{\mathbf{A}}\psi \in \mathcal{D}(D_{\mathbf{A}})\}$ given by the Friedrichs extension. The two components dd^* and d^*d of $H_{\mathbf{A}}$ are unitarily equivalent on the orthogonal complement of $\ker(H_{\mathbf{A}}) = \ker(D_{\mathbf{A}})$. To verify this we note first that, due to the matrix structure of $D_{\mathbf{A}}$, we have

$$(4) \quad \text{sgn}(D_{\mathbf{A}}) := \frac{D_{\mathbf{A}}}{|D_{\mathbf{A}}|} = \begin{pmatrix} 0 & s^* \\ s & 0 \end{pmatrix}$$

on $\ker(D_{\mathbf{A}})^\perp = \ker(d)^\perp \oplus \ker(d^*)^\perp$. Since $\text{sgn}(D_{\mathbf{A}})^2 = \text{Id}$ on $\ker(D_{\mathbf{A}})^\perp$, the maps

$$(5) \quad s : \ker(d)^\perp \rightarrow \ker(d^*)^\perp, \quad s^* : \ker(d^*)^\perp \rightarrow \ker(d)^\perp$$

are unitary and conjugated to each other. By the operator identity $H_{\mathbf{A}} = D_{\mathbf{A}}^2 = \text{sgn}(D_{\mathbf{A}})D_{\mathbf{A}}^2\text{sgn}(D_{\mathbf{A}})$ one concludes that

$$(6) \quad \begin{pmatrix} d^*d & 0 \\ 0 & dd^* \end{pmatrix} \varphi = \begin{pmatrix} s^*dd^*s & 0 \\ 0 & sd^*ds^* \end{pmatrix} \varphi$$

for any $\varphi = (\varphi_1, \varphi_2)^T$ with $\varphi_1 \in \mathcal{D}(d^*d) \cap \ker(d)^\perp$ and $\varphi_2 \in \mathcal{D}(dd^*) \cap \ker(d^*)^\perp$. Hence, on $\ker(d)^\perp$ the operator d^*d is unitarily equivalent to dd^* (considered as an operator on $\ker(d^*)^\perp$). Let us denote the orthogonal projection on $\ker(D_{\mathbf{A}})$ by P_0 and the orthogonal projections on $\ker(d)$, $\ker(d^*)$ by π, π_* . Further, we set

$$P_0^\perp := \mathbb{1} - P_0, \quad \pi^\perp := \mathbb{1} - \pi, \quad \pi_*^\perp := \mathbb{1} - \pi_*.$$

To define our full Hamiltonian let $\mathbf{A} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and $B, V \in C(\mathbb{R}^2, \mathbb{R})$ be such that $B = \text{curl } \mathbf{A}$, then H is given by

$$H\varphi = [D_{\mathbf{A}}^2 + V]\varphi = [(-i\nabla - \mathbf{A})^2 - \sigma_3 B + V]\varphi \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2).$$

In general, the closure of this densely defined operator is not self-adjoint without any restriction on the growth rate of V at infinity. However, there are conditions, very similar to those for the classical Schrödinger operator, to ensure essential self-adjointness.

Proposition 1. *Let $B, V \in C^1(\mathbb{R}^2, \mathbb{R})$ and $\mathbf{A} \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ with $B = \text{curl } \mathbf{A}$. In addition, assume that V fulfills the lower bound*

$$(7) \quad V(\mathbf{x}) \geq -c|\mathbf{x}|^2 + d, \quad \mathbf{x} \in \mathbb{R}^2,$$

for some constants $c > 0, d \in \mathbb{R}$. Then H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$.

Remark 1. *Following the lines of the proof given in Appendix A, we see that the regularity condition on B, V can be relaxed to $B, V \in C_{loc}^\alpha(\mathbb{R}^2, \mathbb{R})$, i.e. both only need to be locally α -Hölder continuous. By a perturbation argument one can also see that it suffices to assume that B, V are C_{loc}^α outside some compact set $K \subset \mathbb{R}^2$, while inside K they only need to be continuous.*

Remark 2. *The self-adjoint operator given by Proposition 1 is locally compact, i.e. for any characteristic function $\chi_{B_R(0)}$ on the ball $B_R(0)$ with radius R , the operator $\chi_{B_R(0)}(H - i)^{-1}$ is compact.*

Remark 3. *Considering the case $V = 0$, we obtain that $H_{\mathbf{A}}$, dd^* and d^*d are essentially self-adjoint on $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$, respectively on $C_0^\infty(\mathbb{R}^2, \mathbb{C})$.*

Note that (7) is the same lower bound on V as one needs for the (magnetic) Schrödinger operator to ensure the essential self-adjointness, whereas no restriction on the growth of B is necessary. The regularity conditions on V and \mathbf{A} are quite strong compared to those for the magnetic Schrödinger operator (see [12]). The reason is that due to the lack of a diamagnetic inequality for $H_{\mathbf{A}}$, one uses a direct argument that requires more regularity on the potentials \mathbf{A} and V . The interesting question remains: Could one relax these conditions for the Pauli operator as in the case of the magnetic Schrödinger operator?

3. MAIN RESULTS

In this section we assume that B, V and \mathbf{A} satisfy the conditions of Proposition 1. It is easy to see that in the following results $B, V \in C^1(\mathbb{R}^2, \mathbb{R})$ can be relaxed to hold only outside some compact set $K \subset \mathbb{R}^2$ as in Remark 1.

Theorem 1. *Assume that*

$$(8) \quad V(\mathbf{x}) \longrightarrow -\infty \quad \text{as} \quad |\mathbf{x}| \rightarrow \infty,$$

$$(9) \quad \left| \frac{\nabla V(\mathbf{x})}{V(\mathbf{x})} \right| \longrightarrow 0 \quad \text{as} \quad |\mathbf{x}| \rightarrow \infty,$$

$$(10) \quad \limsup_{|\mathbf{x}| \rightarrow \infty} \left| \frac{V(\mathbf{x})}{2B(\mathbf{x})} \right| < 1.$$

Then $\sigma_{\text{ess}}(H) = \emptyset$, i.e. H has purely discrete spectrum.

Condition (9) is a restriction on the growth rate of V and rather of technical necessity. The interplay between B and V (as mentioned in the introduction) is described by condition (10). Thus, it is worthwhile to investigate further the dependence of $\sigma(H)$ on this quotient:

One can easily observe that if the quotient of (10) surpasses the constant 1, the spectrum of H changes its character. To see this pick $\Omega \in \ker(d^*d)$, then

$$(dd^* + V)\Omega = (d^*d + 2B + V)\Omega \approx 0$$

if $2B \approx -V$. Therefore, if $\ker(d^*d)$ contains enough functions (which is the case for fields B bounded from below by some positive constant), we obtain points in the essential spectrum of H . One can even show that the condition $2B \approx -V$ (at infinity) does not need to hold globally for obtaining $\sigma_{\text{ess}}(H) \neq \emptyset$. We demonstrate this for a certain class of fields B and potentials V .

Definition 1. *A function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ varies with rate $\nu \in [0, 1]$ on a set $X \subset \mathbb{R}^2$ if there is a constant $C > 0$ such that for all $\mathbf{x} \in X$ it holds that*

$$|f(\mathbf{x} + \mathbf{y})| \leq C|f(\mathbf{x})|,$$

whenever $\mathbf{y} \in \mathbb{R}^2$ satisfies $|\mathbf{y}| \leq \frac{1}{2}|\mathbf{x}|^\nu$

Note that functions of the form $f_1(\mathbf{x}) = c|\mathbf{x}|^s$ and $f_2(\mathbf{x}) = c|x_1|^s$, with $c, s \in \mathbb{R}$, vary with any rate $\nu \in [0, 1]$ on $\mathbb{R}^2 \setminus B_1(0)$ and on $\mathbb{R}^2 \setminus [-1, 1] \times \mathbb{R}$ respectively.

Theorem 2. Assume that there is a sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ with $|\mathbf{x}_n| \rightarrow \infty$ as $n \rightarrow \infty$ and constants $k \in \mathbb{N}$, $\varepsilon \in (0, 1)$ such that $|\nabla V|, |\nabla B|$ vary with rate 0 on $(\mathbf{x}_n)_{n \in \mathbb{N}}$, as well as

$$(11) \quad V(\mathbf{x}_n) \rightarrow -\infty,$$

$$(12) \quad \frac{|\nabla B(\mathbf{x}_n)|^2}{|B(\mathbf{x}_n)|^{1-\varepsilon}}, \frac{|\nabla V(\mathbf{x}_n)|^2}{|V(\mathbf{x}_n)|^{1-\varepsilon}} \rightarrow 0,$$

$$(13) \quad V(\mathbf{x}_n) + 2k|B(\mathbf{x}_n)| \rightarrow 0$$

as $n \rightarrow \infty$. Then $0 \in \sigma_{\text{ess}}(H)$.

Let us now consider the case $V \gg B$ at infinity. The next two theorems state that the accumulation of eigenvalues intensifies, creating more points in the essential spectrum and closing spectral gaps.

Theorem 3. Assume that there is a continuous path $\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^2$, with $|\gamma(t)| \rightarrow \infty$ as $t \rightarrow \infty$, and constants $\epsilon > 0$, $\nu \in [0, 1]$ such that $|\nabla V|, |\nabla B|$ vary with rate ν on $\text{Im}(\gamma)$, as well as

$$(14) \quad \frac{V(\gamma(t))}{2|B(\gamma(t))|} \rightarrow -\infty$$

$$(15) \quad \left(\frac{|\nabla B(\gamma(t))|}{|B(\gamma(t))|} + \frac{|\nabla V(\gamma(t))|}{|V(\gamma(t))|} \right) \left(\frac{|V(\gamma(t))|^3}{B^2(\gamma(t))} \right)^{\frac{1+\epsilon}{2}} \rightarrow 0,$$

$$(16) \quad \frac{1}{|\gamma(t)|^{2\nu}} \left(\frac{|V(\gamma(t))|}{B^2(\gamma(t))} \right)^{1+\epsilon} \rightarrow 0$$

as $t \rightarrow \infty$. In addition, suppose that for all $t \in (0, \infty)$ the inequality

$$(17) \quad B_0 \leq |B(\gamma(t))| \leq \alpha \exp \left(\kappa \left| \frac{V(\gamma(t))}{B(\gamma(t))} \right| \right)$$

holds with constants $\alpha, \kappa, B_0 > 0$. Then $\sigma_{\text{ess}}(H) = \mathbb{R}$.

For our main application, potentials of power-like growth (see discussion after the next theorem), condition (16) imposes unsatisfying restrictions on the growth rate of V/B . At least in the case of a constant magnetic field they can be weakened.

Theorem 4. Let $B = B_0 > 0$ and $V \in C^2(\mathbb{R}^2, \mathbb{R})$. Assume that there is a continuous path $\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^2$, with $|\gamma(t)| \rightarrow \infty$ as $t \rightarrow \infty$, and constants $\epsilon > 0$, $\nu \in [0, 1]$ such that the matrix norm of the Hessian matrix $\|\text{Hess}(V)\|_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ varies with rate ν on $\text{Im}(\gamma)$, as well as

$$(18) \quad V(\gamma(t)) \rightarrow -\infty,$$

$$(19) \quad \|\text{Hess}(V)\|_2(\gamma(t)) |V(\gamma(t))|^{1+\epsilon} \rightarrow 0,$$

$$(20) \quad \frac{1}{|\gamma(t)|^{2\nu}} |V(\gamma(t))|^{1+\epsilon} \rightarrow 0$$

as $t \rightarrow \infty$. In addition, let

$$(21) \quad \limsup_{t \rightarrow \infty} \frac{|\nabla V(\gamma(t))|^2}{|V(\gamma(t))|} < (2B_0)^2.$$

Then $\sigma_{\text{ess}}(H) = \mathbb{R}$.

Remark 4. Note that a well-known, basic example for this last theorem is the case of a constant electric field \mathcal{E}_0 in x_1 -direction with the corresponding potential $V(\mathbf{x}) = \mathcal{E}_0 x_1$.

Remark 5. Results similar to that of Theorems 1–4 can be obtained for the magnetic Schrödinger operator with scalar potentials V by using the same techniques as in the proofs of Theorems 1–4.

Finally, we want to discuss some consequences of our results, in particular with respect to spherically symmetric fields B and potentials V , i.e. $B(\mathbf{x}) = b(|\mathbf{x}|)$, $V(\mathbf{x}) = v(|\mathbf{x}|)$ for $\mathbf{x} \in \mathbb{R}$. Using the rotational gauge

$$\mathbf{A}(\mathbf{x}) := \frac{A(r)}{r} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad A(r) = \frac{1}{r} \int_0^r b(s) s ds,$$

with $r = |\mathbf{x}|$, we decompose H in a direct sum of operators on the half-line. More explicitly, there is a unitary map

$$U : L^2(\mathbb{R}^2, \mathbb{C}^2) \rightarrow \bigoplus_{j \in \mathbb{Z}} L^2(\mathbb{R}^+, \mathbb{C}^2; dr)$$

such that $UHU^* = \bigoplus_{j \in \mathbb{Z}} h_j$, with

$$h_j := \begin{pmatrix} -\partial_r^2 + \frac{j^2 - 1/4}{r^2} & 0 \\ 0 & -\partial_r^2 + \frac{(j+1)^2 - 1/4}{r^2} \end{pmatrix} + A^2(r) - \frac{m_j}{r} A(r) + \sigma_3 A'(r) + v(r)$$

on $L^2(\mathbb{R}^+, \mathbb{C}^2; dr)$, where $m_j = j + \frac{1}{2}$ (see e.g. [13]). It is easy to verify that if

$$(22) \quad \liminf_{r \rightarrow \infty} b(r) > 0,$$

$$(23) \quad A'(r)/A^2(r) \rightarrow 0 \quad \text{as } r \rightarrow \infty,$$

$$(24) \quad \limsup_{r \rightarrow \infty} |v(r)|/A^2(r) < 1,$$

then h_j has purely discrete spectrum for every $j \in \mathbb{Z}$. As a consequence, one can use the relations

$$\sigma_{\#}(H) = \overline{\bigcup_{j \in \mathbb{Z}} \sigma_{\#}(h_j)}, \quad \# \in \{\text{ac}, \text{sc}, \text{pp}\}$$

to conclude that $\sigma(H) = \sigma_{\text{pp}}(H)$, $\sigma_{\text{ac}}(H) = \sigma_{\text{sc}}(H) = \emptyset$ if (22)–(24) are satisfied. To get more information on $\sigma(H) = \sigma_{\text{pp}}(H)$, we employ Theorems 1–4 and obtain:

Corollary 1. Let $b(r) = b_0 r^s$, $v(r) = v_0 r^t$ with $v_0 < 0 < b_0$ and exponents $0 \leq s$, $0 \leq t \leq 2$. Then

- a) $\sigma(H)$ is purely discrete if $0 < t < s$ or $0 < t = s$ and $|v_0| < 2b_0$,
- b) $0 \in \sigma_{\text{ess}}(H)$ if $0 < t = s$ and $|v_0| = 2kB_0$ for some $k \in \mathbb{N}$,
- c) $\sigma(H) = \mathbb{R}$ is dense pure point if $3s < 3t < 2(s+1)$,
- d) $\sigma(H) = \mathbb{R}$ is dense pure point if $s = 0$ and $0 < t < 1$.

The origins of the strong restrictions on s, t in c), d) can easily be tracked back to conditions (15), (16) of Theorem 3 and (19) of Theorem 4. Unfortunately, even in the case of a constant magnetic field ($s = 0$) we cannot cover the full range of potentials ($0 < t \leq 2$) for which one might expect $\sigma(H) = \mathbb{R}$.

4. PROOF OF THEOREM 1

Note that the assumptions imply that either $B(\mathbf{x}) \rightarrow \infty$ or $B(\mathbf{x}) \rightarrow -\infty$. It suffices to consider the case $B(\mathbf{x}) \rightarrow \infty$ as $|\mathbf{x}| \rightarrow \infty$ since otherwise we only have to interchange the roles of d and d^* in the proof. By modifying B and V on a compact set and comparing the corresponding resolvents, we may assume that B and V satisfy

$$(25) \quad V(\mathbf{x}) \leq -1/\delta,$$

$$(26) \quad |\nabla V(\mathbf{x})| \leq |\delta V(\mathbf{x})|,$$

$$(27) \quad |V(\mathbf{x})| \leq 2(1-\eta)B(\mathbf{x}),$$

where $\delta \in (0, \frac{1}{4})$ is fixed, but can be chosen arbitrarily small, and $\eta \in (0, 1)$ is a fixed (δ -independent) constant (c.f. [10] Appendix B).

Using the commutator relation (3), we see that

$$(28) \quad dd^* \geq 2B \geq (1-\eta)^{-1}|V| \geq (1-\eta)^{-1}\delta^{-1}$$

on $C_0^\infty(\mathbb{R}^2, \mathbb{C})$ and therefore on $\mathcal{D}(dd^*)$. Since dd^* and d^*d are isospectral away from 0, we obtain a spectral gap $(0, \beta) \subset \varrho(H_{\mathbf{A}})$, with $\beta = (1-\eta)^{-1}\delta^{-1}$. Thus, 0 can be regarded as an isolated point of the spectrum, which is used in the following commutator estimates.

Lemma 1. *Let $V \in C^1(\mathbb{R}^2, \mathbb{R})$, $B \in C(\mathbb{R}^2, \mathbb{R})$ and $\mathbf{A} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ with $B = \text{curl } \mathbf{A}$. Assume further that the conditions (25)–(27) are fulfilled for $\delta \in (0, \frac{1}{4})$ and $\eta \in (0, 1)$. Then:*

- a) *The operators $[P_0^\perp, V^{-1}]V$, $V[P_0^\perp, V^{-1}]$ are well-defined on $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$ and extend to bounded operators on $L^2(\mathbb{R}^2, \mathbb{C}^2)$ with*

$$\|V[P_0^\perp, V^{-1}]\|, \|[P_0^\perp, V^{-1}]V\| \leq 4\delta^{\frac{3}{2}}.$$

The same holds true if we replace P_0^\perp above by P_0 .

- b) $P_0\mathcal{D}(V), P_0^\perp\mathcal{D}(V) \subset \mathcal{D}(V)$.

Lemma 2. *Let $V \in C^1(\mathbb{R}^2, \mathbb{R})$, $B \in C(\mathbb{R}^2, \mathbb{R})$ and $\mathbf{A} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ with $B = \text{curl } \mathbf{A}$. Assume further that the conditions (25)–(27) are fulfilled for $\delta \in (0, \frac{1}{4})$ and $\eta \in (0, 1)$. Then $[\text{sgn}(D_{\mathbf{A}})P_0^\perp, V^{-1}]$ maps $L^2(\mathbb{R}^2, \mathbb{C}^2)$ into $\mathcal{D}(V)$ and*

$$(29) \quad \|V[\text{sgn}(D_{\mathbf{A}})P_0^\perp, V^{-1}]\| \leq 4\delta^{\frac{3}{2}}.$$

The proofs of these commutator estimates can be found in [10]. Since $D_{\mathbf{A}}$ is a first-order operator, it is much more convenient to commute V with functions of $D_{\mathbf{A}}$ instead of with functions of $H_{\mathbf{A}}$. For proving Theorem 1, it suffices to find a constant $c > 0$ such that

$$(30) \quad \|H\varphi\| \geq c\|V\varphi\|, \quad \varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$$

holds (see Lemma 4 in the appendix).

Proof of Theorem 1. Let $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$. By Lemma 1 we can split $\|H\varphi\|$ as

$$\begin{aligned} \|(H_{\mathbf{A}} + V)\varphi\|^2 &= \|(H_{\mathbf{A}} + V)(P_0 + P_0^\perp)\varphi\|^2 \\ &= \|(VP_0 + (H_{\mathbf{A}} + V)P_0^\perp)\varphi\|^2 \\ &= \|(H_{\mathbf{A}} + V)P_0^\perp\varphi\|^2 + 2\operatorname{Re}\langle (H_{\mathbf{A}} + V)P_0^\perp\varphi, VP_0\varphi \rangle + \|VP_0\varphi\|^2 \\ &= \|(H_{\mathbf{A}} + V)P_0^\perp\varphi\|^2 - \delta\|VP_0^\perp\varphi\|^2 \\ &\quad + 2\operatorname{Re}\langle VP_0\varphi, H_{\mathbf{A}}P_0^\perp\varphi \rangle + \|V\varphi\|^2 - (1 - \delta)\|VP_0^\perp\varphi\|^2. \end{aligned}$$

For the cross-term, condition (26) yields

$$\begin{aligned} |\langle VP_0\varphi, H_{\mathbf{A}}P_0^\perp\varphi \rangle| &= |\langle D_{\mathbf{A}}VP_0\varphi, D_{\mathbf{A}}P_0^\perp\varphi \rangle| \\ &= |\langle [D_{\mathbf{A}}, V]V^{-1}VP_0\varphi, D_{\mathbf{A}}P_0^\perp\varphi \rangle| \\ (31) \quad &\leq \frac{1}{2}\delta^{-\frac{1}{2}}\|(-i\sigma\nabla V)V^{-1}VP_0\varphi\|^2 + \frac{1}{2}\delta^{\frac{1}{2}}\|D_{\mathbf{A}}P_0^\perp\varphi\|^2 \\ &\leq \frac{1}{2}\delta^{\frac{3}{2}}\|VP_0\varphi\|^2 + \frac{1}{2}\delta^{\frac{1}{2}}\|H_{\mathbf{A}}P_0^\perp\varphi\|\|\varphi\| \\ &\leq \frac{1}{4}\delta^{\frac{3}{2}}\|H_{\mathbf{A}}P_0^\perp\varphi\|^2 + \frac{1}{4}\delta^{\frac{3}{2}}(\|V\varphi\|^2 + 2\|VP_0\varphi\|^2). \end{aligned}$$

Applying Lemma 1 a) results in

$$\|VP_0^\perp\varphi\|, \|VP_0\varphi\| \leq (1 + 4\delta^{\frac{3}{2}})\|V\varphi\|,$$

and therefore

$$\begin{aligned} (32) \quad \|V\varphi\|^2 - (1 - \delta)\|VP_0^\perp\varphi\|^2 - \frac{1}{4}\delta^{\frac{3}{2}}\|V\varphi\|^2 - \frac{1}{2}\delta^{\frac{3}{2}}\|VP_0\varphi\|^2 \\ \geq (\delta - 14\delta^{\frac{3}{2}})\|V\varphi\|^2. \end{aligned}$$

Because

$$\begin{aligned} \|(H_{\mathbf{A}} + V)P_0^\perp\varphi\|^2 - \delta^{\frac{3}{2}}\|H_{\mathbf{A}}P_0^\perp\varphi\|^2 - \delta\|VP_0\varphi\|^2 \\ \geq (1 - \varepsilon - \delta^{\frac{3}{2}})\|H_{\mathbf{A}}P_0^\perp\varphi\|^2 + (1 - \varepsilon^{-1} - \delta)\|VP_0^\perp\varphi\|^2 \end{aligned}$$

for any $\varepsilon \in (0, 1)$, it suffices to show, in view of (31) and (32), that

$$(33) \quad \langle H_{\mathbf{A}}P_0^\perp\varphi, H_{\mathbf{A}}P_0^\perp\varphi \rangle + \frac{1 - \varepsilon^{-1} - \delta}{1 - \varepsilon - \delta^{\frac{3}{2}}} \langle VP_0^\perp\varphi, VP_0^\perp\varphi \rangle \geq 0$$

for $\delta > 0$ small enough and some $\varepsilon \in (0, 1)$. We choose $\varepsilon = 1 - \delta^{\frac{1}{2}}$, then

$$-\frac{1 - \varepsilon^{-1} - \delta}{1 - \varepsilon - \delta^{\frac{3}{2}}} = \frac{1}{1 - \delta} \left(\frac{1}{1 - \delta^{\frac{1}{2}}} + \delta^{\frac{1}{2}} \right) =: c_\delta > 0.$$

Since $dd^* \geq 2B$ and therefore $\ker(d^*) = \{0\}$, we have

$$(34) \quad P_0^\perp = \begin{pmatrix} \pi^\perp & 0 \\ 0 & \pi_*^\perp \end{pmatrix} = \begin{pmatrix} \pi^\perp & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

Setting $\varphi = (\varphi_1, \varphi_2)^T$, one can rewrite (33) as

$$\|H_{\mathbf{A}}P_0^\perp\varphi\|^2 - c_\delta\|VP_0^\perp\varphi\|^2 = \|d^*d\pi^\perp\varphi_1\|^2 - c_\delta\|V\pi^\perp\varphi_1\|^2 + \|dd^*\varphi_2\|^2 - c_\delta\|V\varphi_2\|^2.$$

By using the isometries s, s^* given in (5), relation (6) and estimate (28), one obtains

$$\begin{aligned}
\|dd^*\varphi_2\|^2 - c_\delta\|V\varphi_2\|^2 &= \langle d^*\varphi_2, d^*dd^*\varphi_2 \rangle - c_\delta\langle \sqrt{-V}\varphi_2, |V|\sqrt{-V}\varphi_2 \rangle \\
&= \langle sd^*\varphi_2, dd^*sd^*\varphi_2 \rangle - c_\delta\langle \sqrt{-V}\varphi_2, |V|\sqrt{-V}\varphi_2 \rangle \\
&\geq \langle sd^*\varphi_2, 2Bsd^*\varphi_2 \rangle - c_\delta\langle d^*\sqrt{-V}\varphi_2, d^*\sqrt{-V}\varphi_2 \rangle \\
&\geq \langle sd^*\varphi_2, 2Bsd^*\varphi_2 \rangle - c_\delta\langle \sqrt{-V}d^*\varphi_2, \sqrt{-V}d^*\varphi_2 \rangle \\
&\quad - c_\delta\langle [d^*, \sqrt{-V}]\varphi_2, \sqrt{-V}d^*\varphi_2 \rangle \\
&\quad - c_\delta\langle \sqrt{-V}d^*\varphi_2, [d^*, \sqrt{-V}]\varphi_2 \rangle \\
&\quad - c_\delta\langle [d^*, \sqrt{-V}]\varphi_2, [d^*, \sqrt{-V}]\varphi_2 \rangle \\
&\geq \|\sqrt{2B}sd^*\varphi_2\|^2 - c_\delta\left(\|[d^*, \sqrt{-V}]\varphi_2\| + \|\sqrt{-V}d^*\varphi_2\|\right)^2 \\
&\geq \|\sqrt{2B}sd^*\varphi_2\|^2 - c_\delta\left(\delta\|sd^*\varphi_2\| + (1 + 4\delta^{\frac{3}{2}})\|\sqrt{-V}sd^*\varphi_2\|\right)^2 \\
(35) \quad &\geq [1 - c_\delta(1 - \eta)(1 + 15\delta^{\frac{3}{2}})]\|\sqrt{2B}sd^*\varphi_2\|^2,
\end{aligned}$$

where we applied the bound $\|\sqrt{-V}[s\pi^\perp, \sqrt{-V}^{-1}]\| \leq 4\delta^2$. For the latter we write

$$\sqrt{-V}\left[\text{sgn}(D_{\mathbf{A}})P_0^\perp, \sqrt{-V}^{-1}\right] = \begin{pmatrix} 0 & \sqrt{-V}\left[s^*, \sqrt{-V}^{-1}\right] \\ \sqrt{-V}\left[s\pi^\perp, \sqrt{-V}^{-1}\right] & 0 \end{pmatrix}$$

and therefore, by Lemma 2 with $\sqrt{-V}$ instead of V , we get

$$\left\|\sqrt{-V}\left[s\pi^\perp, \sqrt{-V}^{-1}\right]\right\| \leq \left\|\sqrt{-V}\left[\text{sgn}(D_{\mathbf{A}})P_0^\perp, \sqrt{-V}^{-1}\right]\right\| \leq 4\delta^{\frac{3}{2}}.$$

Similarly, we obtain a lower bound for $\|d^*d\pi^\perp\varphi_1\|^2 - c_\delta\|V\pi^\perp\varphi_1\|^2$ by using again the upper relation of Equation (6). More precisely,

$$\begin{aligned}
\|d^*d\pi^\perp\varphi_1\|^2 - c_\delta\|V\pi^\perp\varphi_1\|^2 &= \|d^*ds^*s\pi^\perp\varphi_1\|^2 - c_\delta\|Vs^*s\pi^\perp\varphi_1\|^2 \\
&= \|dd^*s\pi^\perp\varphi_1\|^2 - c_\delta\|Vs^*V^{-1}Vs\pi^\perp\varphi_1\|^2 \\
&= \|dd^*s\pi^\perp\varphi_1\|^2 - c_\delta\|(s^* + V[s^*, V^{-1}])Vs\pi^\perp\varphi_1\|^2,
\end{aligned}$$

where $V[s^*, V^{-1}]$ is one of the components of the operator

$$V\left[\text{sgn}(D_{\mathbf{A}})P_0^\perp, V^{-1}\right] = \begin{pmatrix} 0 & V[s^*, V^{-1}] \\ V[s\pi^\perp, V^{-1}] & 0 \end{pmatrix},$$

so Lemma 2 yields $\|V[s^*, V^{-1}]\| \leq 4\delta^{\frac{3}{2}}$. Thus,

$$\|d^*d\pi^\perp\varphi_1\|^2 - c_\delta\|V\pi^\perp\varphi_1\|^2 \geq \|dd^*s\pi^\perp\varphi_1\|^2 - c_\delta(1 + 10\delta^{\frac{3}{2}})\|Vs\pi^\perp\varphi_1\|^2.$$

We note that $s\pi^\perp\varphi_1 \subset \mathcal{D}(dd^*) \subset \mathcal{D}(V)$, hence we can use (35) (by approximating $s\pi^\perp\varphi_1$ through C_0^∞ -functions in the graph norm of dd^*) to conclude that

$$\begin{aligned}
&\|d^*d\pi^\perp\varphi_1\|^2 - c_\delta\|V\pi^\perp\varphi_1\|^2 \\
&\geq [1 - c_\delta(1 - \eta)(1 + 10\delta^{\frac{3}{2}})(1 + 15\delta^{\frac{3}{2}})]\|\sqrt{2B}sd^*s\pi^\perp\varphi_1\|^2.
\end{aligned}$$

Combining this inequality with (35) leads to

$$\begin{aligned} & \|H_{\mathbf{A}} P_0^\perp \varphi\|^2 - c_\delta \|V P_0^\perp \varphi\|^2 \\ & \geq [1 - c_\delta(1 - \eta)(1 + 50\delta^{\frac{3}{2}})] \left(\|\sqrt{2B} s d^* \varphi_2\|^2 + \|\sqrt{2B} d \pi^\perp \varphi_1\|^2 \right), \end{aligned}$$

where the r.h.s is non-negativ for δ small enough. \square

5. PROOFS OF THEOREM 2 AND 3

The basic strategy of the proofs is to represent B and V locally through constant values $V_n := V(\mathbf{x}_n)$ and $B_n := B(\mathbf{x}_n)$ along a sequence $(\mathbf{x}_n)_{n \in \mathbb{N}} \subset \mathbb{R}^2$. Since one also needs to compare vector potentials associated to B_n and B , we use the gauges

$$\begin{aligned} \mathbf{A}_n(\mathbf{x}) &:= \int_0^1 B_n \wedge (\mathbf{x} - \mathbf{x}_n) s ds = \frac{1}{2} B_n \wedge (\mathbf{x} - \mathbf{x}_n), \\ \tilde{\mathbf{A}}_n(\mathbf{x}) &:= \int_0^1 B(\mathbf{x}_n + s(\mathbf{x} - \mathbf{x}_n)) \wedge (\mathbf{x} - \mathbf{x}_n) s ds, \end{aligned}$$

where $a \wedge \mathbf{v} := a(-v_2, v_1)$ for $a \in \mathbb{R}$ and $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$. The two given vector potentials satisfy $\text{curl } \mathbf{A}_n = \text{curl } \tilde{\mathbf{A}}_n = B$, hence for every $n \in \mathbb{N}$ there exists a function $g_n \in C^2(\mathbb{R}^2, \mathbb{R})$ such that $\nabla g_n = \mathbf{A} - \tilde{\mathbf{A}}_n$. In addition, for every vector potential \mathbf{A}_n , representing the constant magnetic fields B_n , we obtain operators d_n and d_n^* , $n \in \mathbb{N}$, defined as in (2). For a sequence of natural numbers $(k_n)_{n \in \mathbb{N}}$ we set

$$\psi_n(\mathbf{x}) := \begin{pmatrix} (d_n^*)^{k_n} e^{-B_n |\mathbf{x} - \mathbf{x}_n|^2 / 4} \\ 0 \end{pmatrix}.$$

Iterating commutator relation (3) for d_n, d_n^* yields

$$(36) \quad d_n^* d_n [(d_n^*)^{k_n} e^{-B_n |\mathbf{x} - \mathbf{x}_n|^2 / 4}] = 2k_n B_n [(d_n^*)^{k_n} e^{-B_n |\mathbf{x} - \mathbf{x}_n|^2 / 4}], \quad n \in \mathbb{N},$$

i.e. ψ_n is an eigenfunction of $H_{\mathbf{A}_n}$ with the corresponding eigenvalue $2k_n B_n$. For the localization let $\chi \in C_0^\infty(\mathbb{R}^2, [0, 1])$ be such that $\chi(\mathbf{x}) = 1$ for $|\mathbf{x}| \leq 1$ and $\chi(\mathbf{x}) = 0$ for $|\mathbf{x}| \geq 2$. We set

$$\chi_n(\mathbf{x}) := \chi\left(\frac{\mathbf{x} - \mathbf{x}_n}{r_n}\right),$$

where the $r_n > 0$ will be chosen in the proofs. For the Weyl sequence we define the functions φ_n through

$$(37) \quad \varphi_n(\mathbf{x}) := e^{i g_n(\mathbf{x})} \chi_n(\mathbf{x}) \psi_n(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2,$$

with $n \in \mathbb{N}$. Bounds on the norm of φ_n can be obtained as in [10]. They are given by:

Lemma 3. *For all $n \in \mathbb{N}$ large enough we have*

$$(38) \quad \|\varphi_n\|^2 \leq \|\psi_n\|^2 = 2\pi \int_0^\infty (B_n r)^{2k_n} e^{-\frac{B_n}{2} r^2} dr = 2^{k_n+1} \pi B_n^{k_n-1} k_n!,$$

$$(39) \quad \|\varphi_n\|^2 \geq \|\psi_n\|^2 \left(1 - \frac{1}{k_n!} \int_{\frac{1}{2} B_n r_n^2}^\infty s^{k_n} e^{-s} ds \right).$$

Now $H\varphi_n$ can be written as

$$\begin{aligned}
 e^{-ig_n}(H_{\mathbf{A}} + V)\varphi_n &= (H_{\tilde{\mathbf{A}}_n} + V)\chi_n\psi_n \\
 &= (H_{\mathbf{A}_n} + V)\chi_n\psi_n + 2(\tilde{\mathbf{A}}_n - \mathbf{A}_n)(-i\nabla - \mathbf{A}_n)\chi_n\psi_n + \\
 (40) \quad &(\tilde{\mathbf{A}}_n - \mathbf{A}_n)^2\chi_n\psi_n - i\nabla \cdot (\tilde{\mathbf{A}}_n - \mathbf{A}_n)\chi_n\psi_n + \\
 &(B - B_n)\chi_n\psi_n,
 \end{aligned}$$

with the localization error

$$\begin{aligned}
 (41) \quad &(H_{\mathbf{A}_n} + V)\chi_n\psi_n - \chi_n(H_{\mathbf{A}_n} + V)\psi_n \\
 &= -(\Delta\chi_n)\psi_n + 2(-i\nabla\chi_n)(-i\nabla - \mathbf{A}_n)\psi_n.
 \end{aligned}$$

To prove Theorem 2 and Theorem 3, we estimate each term of (40) separately. For the proofs we use the notation $K_n := \{x \in \mathbb{R}^2 \mid r_n \leq |\mathbf{x} - \mathbf{x}_n| \leq 2r_n\}$ with $n \in \mathbb{N}$.

Proof of Theorem 2. We set $k_n = k$ and choose the radii to be $r_n^{-4} = B_n^{(2-\epsilon)}$. Then, for any $p \geq 0$,

$$\frac{(B_n)^p}{k_n!} \int_{\frac{1}{2}B_n r_n^2}^{\infty} s^{k_n} e^{-s} ds = \frac{(B_n)^p}{k!} \int_{\frac{1}{2}B_n^{\epsilon/2}}^{\infty} s^k e^{-s} ds \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Further, we have $\|\psi_n\|^2 \leq 2\|\varphi_n\|^2$ for $n \in \mathbb{N}$ large enough. For treating the terms on the r.h.s. of (40), we estimate

$$\begin{aligned}
 &\|(\tilde{\mathbf{A}}_n - \mathbf{A}_n)(-i\nabla - \mathbf{A}_n)\chi_n\psi_n\|^2 \\
 &\leq C_1 r_n^4 |\nabla B(\mathbf{x}_n)|^2 \|(-i\nabla - \mathbf{A}_n)\chi_n\psi_n\|^2 \\
 &\leq 2C_1 r_n^4 |\nabla B(\mathbf{x}_n)|^2 \left[(2k+1)B_n \|\psi_n\|^2 + r_n^{-2} \|\nabla\chi\|_{\infty}^2 \int_{K_n} |\psi_n(\mathbf{x})|^2 d^2x \right] \\
 &\leq 16kC_1 B_n \frac{|\nabla B(\mathbf{x}_n)|^2}{B_n^{2-\epsilon}} \|\psi_n\|^2 + 4C_1 \|\nabla\chi\|_{\infty}^2 \|\psi_n\|^2 B_n^{(2-\epsilon)/2} \frac{1}{k!} \int_{\frac{1}{2}B_n^{\epsilon/2}}^{\infty} s^k e^{-s} ds.
 \end{aligned}$$

In addition,

$$\begin{aligned}
 \|(\tilde{\mathbf{A}}_n - \mathbf{A}_n)^2\chi_n\psi_n\|^2 &\leq C_2 r_n^4 |\nabla B(\mathbf{x}_n)|^4 \|\psi_n\|^2 \leq C_2 \frac{|\nabla B(\mathbf{x}_n)|^4}{B_n^{2(1-\epsilon)}} B_n^{-\epsilon} \|\psi_n\|^2, \\
 \|\operatorname{div}(\tilde{\mathbf{A}}_n - \mathbf{A}_n)\chi_n\psi_n\|^2 &= \|\operatorname{div}\tilde{\mathbf{A}}_n\chi_n\psi_n\|^2 \leq C_3 r_n^2 |\nabla B(\mathbf{x}_n)|^2 \|\psi_n\|^2 \\
 &\leq C_3 \frac{|\nabla B(\mathbf{x}_n)|^2}{B_n^{1-\epsilon}} B_n^{-\epsilon/2} \|\psi_n\|^2, \\
 \|(B - B_n)\chi_n\psi_n\|^2 &\leq C_4 r_n^2 |\nabla B(\mathbf{x}_n)|^2 \|\psi_n\|^2 \leq C_4 \frac{|\nabla B(\mathbf{x}_n)|^2}{B_n^{1-\epsilon}} B_n^{-\epsilon/2} \|\psi_n\|^2.
 \end{aligned}$$

For the first term of the r.h.s of (40) we get, due to (41),

$$\begin{aligned}
 &\|(H_{\mathbf{A}_n} + V)\chi_n\psi_n\| \\
 &\leq \|\chi_n(H_{\mathbf{A}_n} + V)\psi_n\| + \|(\Delta\chi_n)\psi_n\| + 2\|(-i\nabla\chi_n)(-i\nabla - \mathbf{A}_n)\psi_n\|,
 \end{aligned}$$

with

$$\begin{aligned}
 \|(\Delta\chi_n)\psi_n\|^2 &\leq r_n^{-4} \|\Delta\chi\|_{\infty}^2 \int_{K_n} |\psi_n(\mathbf{x})|^2 d^2x \\
 &\leq 2\|\Delta\chi\|_{\infty}^2 \|\psi_n\|^2 \frac{1}{k!} B_n^{2-\epsilon} \int_{\frac{1}{2}B_n^{\epsilon/2}}^{\infty} s^k e^{-s} ds,
 \end{aligned}$$

and

$$\begin{aligned}
\|(-i\nabla\chi_n)(-i\nabla - \mathbf{A}_n)\psi_n\|^2 &\leq r_n^{-2}\|\nabla\chi\|_\infty^2 \int_{K_n} |(-i\nabla - \mathbf{A}_n)\psi_n(\mathbf{x})|^2 d^2x \\
&\leq \|\nabla\chi\|_\infty^2 B_n^{(2-\epsilon)/2} (2k+1) B_n \int_{K_n} |\psi_n(\mathbf{x})|^2 d^2x \\
&\leq \|\nabla\chi\|_\infty^2 \|\psi_n\|^2 (2k+1) B_n^{2-\epsilon/2} \int_{\frac{1}{2}B_n^{\epsilon/2}}^\infty s^k e^{-s} ds.
\end{aligned}$$

Because of (36) and since $|\nabla V|$ vary with rate 0, we conclude by the mean value theorem that

$$\begin{aligned}
\|\chi_n(H_{\mathbf{A}_n} + V)\psi_n\|^2 &\leq \|\chi_n(V + 2kB_n)\psi_n\|^2 \\
&\leq C_5 |\nabla V(\mathbf{x}_n)|^2 r_n^2 \|\chi_n\psi_n\|^2 + (2kB_n + V_n)^2 \|\chi_n\psi_n\|^2 \\
&\leq C_5 (4k)^{1-\epsilon} \frac{|\nabla V(\mathbf{x}_n)|^2}{|V_n|^{1-\epsilon}} \|\varphi_n\|^2 + (2kB_n + V_n)^2 \|\varphi_n\|^2.
\end{aligned}$$

Hence, by (40) and conditions (11)–(13), we see that $\|(H_{\mathbf{A}} + V)\varphi_n\|/\|\varphi_n\| \rightarrow 0$ as $n \rightarrow \infty$. In addition, note that $r_n \rightarrow 0$ as $n \rightarrow \infty$, so we can assume that the φ_n have mutually disjoint support, i.e. $(\varphi_n)_{n \in \mathbb{N}}$ is a Weyl sequence for 0. \square

Proof of Theorem 3. We first note that it suffices to proof $0 \in \sigma_{\text{ess}}(H)$ since for $E \in \mathbb{R}$ we consider $V_E := V - E$ instead of V , which also fulfills (14)–(17) along γ . Because $\mathbb{R}^+ \ni t \mapsto V(\gamma(t))/B((\gamma(t)))$ is continuous and (14) holds, we find points $(\mathbf{x}_n)_{n \in \mathbb{N}} \subset \text{Im}(\gamma)$, with $|\mathbf{x}_n| \rightarrow \infty$ as $n \rightarrow \infty$, such that $2nB(\mathbf{x}_n) = -V(\mathbf{x}_n)$. We choose $k_n = n$ and set

$$(42) \quad r_n := \sqrt{2n^{1+\epsilon}/B_n}.$$

Note that $r_n/|\mathbf{x}_n|^\nu \rightarrow 0$ as $n \rightarrow \infty$ by (16). In particular, we might assume the φ_n 's to have mutually disjoint support. Further, for any $\lambda \geq 0$,

$$\frac{e^{\lambda n}}{n!} \int_{n^{1+\epsilon}}^\infty s^n e^{-s} ds \leq e^{\lambda n} \exp(n \ln(2n) - n^{1+\epsilon}/2 + n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, we can choose $N \in \mathbb{N}$ so large that $\|\varphi_n\|^2 \leq \|\psi_n\|^2 \leq 2\|\varphi_n\|^2$ for $n \geq N$. Proceeding as in the proof of Theorem 2, we obtain

$$\begin{aligned}
&\|(\tilde{\mathbf{A}}_n - \mathbf{A}_n)(-i\nabla - \mathbf{A}_n)\chi_n\psi_n\|^2 \\
&\leq C_6 r_n^4 |\nabla B(\mathbf{x}_n)|^2 (2n+1) B_n \left[\|\psi_n\|^2 + r_n^{-2} \|\nabla\chi\|_\infty^2 \int_{K_n} |\psi_n(\mathbf{x})|^2 d^2x \right] \\
&\leq 16C_6 n^{3+2\epsilon} B_n \frac{|\nabla B(\mathbf{x}_n)|^2}{B_n^2} \|\psi_n\|^2 + C_6 \|\nabla\chi\|_\infty^2 \|\psi_n\|^2 B_n \frac{1}{n!} \int_{n^{1+\epsilon}}^\infty s^n e^{-s} ds \\
&\leq \tilde{C}_6 \left(\frac{|V_n|^3}{B_n^2} \right)^{1+\epsilon} \frac{|\nabla B(\mathbf{x}_n)|^2}{B_n^2} \|\psi_n\|^2 + \alpha C_6 \|\nabla\chi\|_\infty^2 \|\psi_n\|^2 \frac{e^{2\kappa n}}{n!} \int_{n^{1+\epsilon}}^\infty s^n e^{-s} ds.
\end{aligned}$$

Using (as in the first inequality above) that $|\nabla B|$ vary with rate ν , we conclude

$$\begin{aligned}
\|(\tilde{\mathbf{A}}_n - \mathbf{A}_n)^2 \chi_n \psi_n\|^2 &\leq C_7 r_n^4 |\nabla B(\mathbf{x}_n)|^4 \|\psi_n\|^2 \leq \frac{C_7}{4} \frac{|V_n|^6}{B_n^4} \frac{|\nabla B(\mathbf{x}_n)|^4}{B_n^4} \|\psi_n\|^2, \\
\|\text{div}(\tilde{\mathbf{A}}_n - \mathbf{A}_n) \chi_n \psi_n\|^2 &\leq C_8 r_n^2 |\nabla B(\mathbf{x}_n)|^2 \|\psi_n\|^2 \leq \frac{C_8}{2} \frac{|V_n|^3}{B_n^2} \frac{|\nabla B(\mathbf{x}_n)|^2}{B_n^2} \|\psi_n\|^2,
\end{aligned}$$

$$\|(B - B_n)\chi_n\psi_n\|^2 \leq C_9 r_n^2 |\nabla B(\mathbf{x}_n)|^2 \|\psi_n\|^2 \leq \frac{C_9}{2} \frac{|V_n|^3}{B_n^2} \frac{|\nabla B(\mathbf{x}_n)|^2}{B_n^2} \|\psi_n\|^2.$$

We estimate, using equality (41), the first term on the r.h.s. of (40) by

$$\begin{aligned} \|(H_{\mathbf{A}_n} + V)\chi_n\psi_n\| \\ \leq \|(V - V_n)\chi_n\psi_n\| + \|(\Delta\chi_n)\psi_n\| + 2\|(-i\nabla\chi_n)(-i\nabla - \mathbf{A}_n)\psi_n\|, \end{aligned}$$

with

$$\begin{aligned} \|(\Delta\chi_n)\psi_n\|^2 &\leq \|\Delta\chi\|_\infty^2 \|\psi_n\|^2 \frac{B_n^2}{n^{2+2\epsilon}} \frac{1}{n!} \int_{n^{1+\epsilon}}^\infty s^n e^{-s} ds \\ &\leq \alpha^2 \|\Delta\chi\|_\infty^2 \|\psi_n\|^2 \frac{e^{4\kappa n}}{n^{2+2\epsilon}} \frac{1}{n!} \int_{n^{1+\epsilon}}^\infty s^n e^{-s} ds, \end{aligned}$$

and

$$\begin{aligned} \|(-i\nabla\chi_n)(-i\nabla - \mathbf{A}_n)\psi_n\|^2 &\leq \|\nabla\chi\|_\infty^2 \frac{B_n}{n^{1+\epsilon}} \int_{K_n} (2n+1) B_n |\psi_n(\mathbf{x})|^2 d^2x \\ &\leq \alpha \|\nabla\chi\|_\infty^2 \|\psi_n\|^2 \frac{e^{2\kappa n}}{n^{1+\epsilon}} \frac{1}{n!} \int_{n^{1+\epsilon}}^\infty s^n e^{-s} ds. \end{aligned}$$

Since $|\nabla V|$ vary with rate ν , we have

$$\begin{aligned} \|\chi_n(H_{\mathbf{A}_n} + V)\psi_n\|^2 &\leq \|\chi_n(V - V_n)\psi_n\|^2 \\ &\leq C_{10} |\nabla V(\mathbf{x}_n)|^2 r_n^2 \|\psi_n\|^2 \leq \tilde{C}_{10} \left(\frac{|V_n|^3}{B_n^2} \right)^{1+\epsilon} \frac{|\nabla V(\mathbf{x}_n)|^2}{|V_n|^2} \|\psi_n\|^2. \end{aligned}$$

We see that $\|(H_{\mathbf{A}_n} + V)\varphi_n\|/\|\varphi_n\| \rightarrow 0$ as $n \rightarrow \infty$ and therefore, by (40) and the estimates above, that $\|(H_{\mathbf{A}} + V)\varphi_n\|/\|\varphi_n\| \rightarrow 0$ as $n \rightarrow \infty$. \square

6. PROOF OF THEOREM 4

Throughout this section we consider the case of a constant magnetic field $B(\mathbf{x}) = B_0$. In addition, we assume that \mathbf{A} is in the rotational gauge, i.e.

$$\mathbf{A}(\mathbf{x}) = \frac{B_0}{2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

Note that $H_{\mathbf{A}}$ is invariant under rotations. More precisely, for a special orthogonal matrix $\mathcal{R} \in SO(2, \mathbb{R})$ define the unitary map

$$U_{\mathcal{R}} : L^2(\mathbb{R}^2, \mathbb{C}^2) \rightarrow L^2(\mathbb{R}^2, \mathbb{C}^2), \quad \psi(\cdot) \mapsto \psi(\mathcal{R}^{-1} \cdot),$$

then $U_{\mathcal{R}}^{-1} H_{\mathbf{A}} U_{\mathcal{R}} = H_{\mathbf{A}}$ and therefore

$$U_{\mathcal{R}}^{-1} (H_{\mathbf{A}} + V) U_{\mathcal{R}} = H_{\mathbf{A}} + V_{\mathcal{R}} \quad \text{with } V_{\mathcal{R}}(\cdot) = V(\mathcal{R} \cdot).$$

To construct a Weyl sequence, consider a second gauge $\tilde{\mathbf{A}}(\mathbf{x}) = B_0 x_1 \hat{e}_2$, called the Landau gauge. Then our Hamiltonian reads

$$\begin{aligned} (43) \quad H_{\tilde{\mathbf{A}}} + V &= -\partial_1^2 + (-i\partial_2 - B_0 x_1)^2 - \sigma_3 B_0 + V \\ &= \tilde{d}^* \tilde{d} + B_0 - \sigma_3 B_0 + V, \end{aligned}$$

with

$$\tilde{d} = -i\partial_1 + i(-i\partial_2 - B_0 x_1), \quad \tilde{d}^* = -i\partial_1 - i(-i\partial_2 - B_0 x_1).$$

For electric fields of the form $V(\mathbf{x}) = V_0 + \mathcal{E}_0(x_1 - \zeta)$, with constants $V_0, \mathcal{E}_0, \zeta \in \mathbb{R}$, we can write

$$\begin{aligned} H_{\tilde{\mathbf{A}}} + V &= -\partial_1^2 + (-i\partial_2 - B_0x_1)^2 - \sigma_3 B_0 + V_0 + \mathcal{E}_0(x_1 - \zeta) \\ (44) \quad &= -\partial_1^2 + B_0^2 \left(x_1 - \frac{1}{B_0}(-i\partial_2 - \frac{\mathcal{E}_0}{2B_0})\right)^2 \\ &\quad + \frac{\mathcal{E}_0}{B_0}(-i\partial_2 - \frac{\mathcal{E}_0}{2B_0}) - \mathcal{E}_0\zeta - \sigma_3 B_0 + V_0 + \left(\frac{\mathcal{E}_0}{2B_0}\right)^2. \end{aligned}$$

Performing a Fourier transform in x_2 , we obtain the direct integral representation

$$H_{\tilde{\mathbf{A}}} + V \cong \int_{\mathbb{R}}^{\oplus} h(\xi) d\xi$$

on $L^2(\mathbb{R}_\xi, L^2(\mathbb{R}, \mathbb{C}^2))$, with

$$\begin{aligned} h(\xi) &= -\partial_1^2 + B_0^2 \left(x_1 - \frac{1}{B_0}(\xi - \frac{\mathcal{E}_0}{2B_0})\right)^2 + \frac{\mathcal{E}_0}{B_0}(\xi - \frac{\mathcal{E}_0}{2B_0}) - \mathcal{E}_0\zeta - \sigma_3 B_0 + V_0 + \left(\frac{\mathcal{E}_0}{2B_0}\right)^2 \\ &= -\partial_1^2 + B_0^2 (x_1 - \hat{\zeta})^2 + \mathcal{E}_0\hat{\zeta} - \mathcal{E}_0\zeta - \sigma_3 B_0 + V_0 + \left(\frac{\mathcal{E}_0}{2B_0}\right)^2. \end{aligned}$$

Here we set $\hat{\zeta} = \frac{1}{B_0}(\xi - \frac{\mathcal{E}_0}{2B_0})$. Note that $h(\xi)$ is the Hamiltonian of a shifted harmonic oscillator. Thus, we define for $n \in \mathbb{N}_0$

$$\phi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \vartheta_n(x) e^{-x^2/2}, \quad x \in \mathbb{R},$$

where ϑ_n denotes the n -th Hermite polynomial. The normalized functions

$$\hat{\psi}_{\mathcal{E}_0, n, \xi}(x_1) := \sqrt[4]{B_0} \begin{pmatrix} \phi_n(\sqrt{B_0}(x_1 - \frac{1}{B_0}(\xi - \frac{\mathcal{E}_0}{2B_0}))) \\ 0 \end{pmatrix}$$

fulfill the equation

$$h(\xi) \hat{\psi}_{\mathcal{E}_0, n, \xi} = \left(2nB_0 + \mathcal{E}_0\left(\frac{1}{B_0}\left(\xi - \frac{\mathcal{E}_0}{2B_0}\right) - \zeta\right) + V_0 + \left(\frac{\mathcal{E}_0}{2B_0}\right)^2\right) \hat{\psi}_{\mathcal{E}_0, n, \xi}.$$

Hence,

$$(45) \quad \psi_{\mathcal{E}_0, n, \xi}(x_1, x_2) := e^{i\xi x_2} \hat{\psi}_{\mathcal{E}_0, n, \xi}(x_1)$$

satisfies

$$(46) \quad [H_{\tilde{\mathbf{A}}} + V] \psi_{\mathcal{E}_0, n, \xi} = \left(2nB_0 + \mathcal{E}_0\left(\frac{1}{B_0}\left(\xi - \frac{\mathcal{E}_0}{2B_0}\right) - \zeta\right) + V_0 + \left(\frac{\mathcal{E}_0}{2B_0}\right)^2\right) \psi_{\mathcal{E}_0, n, \xi}$$

for $\xi \in \mathbb{R}$ and $n \in \mathbb{N}_0$, seen as a differential equation. In addition, we have

$$(47) \quad \tilde{d} \psi_{\mathcal{E}_0, n, \xi} = -i\sqrt{2nB_0} \psi_{\mathcal{E}_0, n-1, \xi} + i\frac{\mathcal{E}_0}{2B_0} \psi_{\mathcal{E}_0, n, \xi},$$

$$(48) \quad \tilde{d}^* \psi_{\mathcal{E}_0, n, \xi} = i\sqrt{2(n+1)B_0} \psi_{\mathcal{E}_0, n+1, \xi} - i\frac{\mathcal{E}_0}{2B_0} \psi_{\mathcal{E}_0, n, \xi}.$$

Proof of Theorem 4. As argued in the proof of Theorem 3, it suffices to find a Weyl sequence for $E = 0$. Because of (18) and (21), there exists a sequence $\{\mathbf{y}_n\}_{n \in \mathbb{N}} \subset \text{Im}(\gamma)$ such that

$$(49) \quad V(\mathbf{y}_n) = -2nB_0 - \left(\frac{|\nabla V(\mathbf{y}_n)|}{2B_0}\right)^2.$$

Further, there are rotations $\mathcal{R}_n \in SO(2, \mathbb{R})$ such that $\nabla V_{\mathcal{R}_n}(\mathbf{x}_n) = |\nabla V_{\mathcal{R}_n}(\mathbf{x}_n)| \hat{e}_1$, with $\mathbf{x}_n = \mathcal{R}_n^{-1} \mathbf{y}_n = (x_{n,1}, x_{n,2})^T$ for $n \in \mathbb{N}$. We set

$$(50) \quad V_n := V(\mathbf{y}_n) = V_{\mathcal{R}_n}(\mathbf{x}_n),$$

$$(51) \quad \mathcal{E}_n := |\nabla V(\mathbf{y}_n)| = |\nabla V_{\mathcal{R}_n}(\mathbf{x}_n)|,$$

$$(52) \quad \xi_n := B_0 x_{n,1} + \frac{\varepsilon_n}{2B_0}.$$

For the Weyl functions let $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$ with $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Define

$$\chi_{n,j}(x) := \chi\left(\frac{x - x_{n,j}}{r_n}\right),$$

for $j = 1, 2$, and

$$\begin{aligned} \varphi_n(\mathbf{x}) &:= \chi_{n,1}(x_1)\chi_{n,2}(x_2)\psi_{\varepsilon_n, n, \xi_n}(x_1, x_2) \\ &= \chi\left(\frac{x_2 - x_{n,2}}{r_n}\right) e^{-i\xi_n x_2} \chi\left(\frac{x_1 - x_{n,1}}{r_n}\right) \begin{pmatrix} \sqrt[4]{B_0} \phi_n(\sqrt{B_0}(x_1 - x_{n,1})) \\ 0 \end{pmatrix}, \end{aligned}$$

where the localization radii r_n are chosen to be $r_n := \sqrt{n^{1+\epsilon}/B_0}$. Note that

$$(53) \quad r_n \leq 2r_n \int_{-\sqrt{n^{1+\epsilon}}}^{\sqrt{n^{1+\epsilon}}} |\phi_n(x)|^2 dx \leq \|\varphi_n\|^2 \leq 4r_n \int_{-2\sqrt{n^{1+\epsilon}}}^{2\sqrt{n^{1+\epsilon}}} |\phi_n(x)|^2 dx \leq 4r_n$$

for $n \in \mathbb{N}$ large enough (see Lemma 5 in the appendix). By denoting $g(\mathbf{x}) = \frac{B_0}{2} x_1 x_2$ for $\mathbf{x} \in \mathbb{R}^2$, we get, due to (43), (46), (49) and (52), that

$$\begin{aligned} (54) \quad HU_{\mathcal{R}_n} e^{-ig} \varphi_n &= U_{\mathcal{R}_n} e^{-ig} [H_{\tilde{\mathbf{A}}} + V_{\mathcal{R}_n}] \varphi_n \\ &= U_{\mathcal{R}_n} e^{-ig} (\tilde{d}^* \tilde{d} \varphi_n - \chi_{n,1} \chi_{n,2} \tilde{d}^* \tilde{d} \psi_{\varepsilon_n, n, \xi_n}) + \\ &\quad U_{\mathcal{R}_n} e^{-ig} [V_{\mathcal{R}_n} - V_n - \varepsilon_n(x_1 - x_{1,n})] \varphi_n. \end{aligned}$$

The localization error results in

$$\begin{aligned} \tilde{d}^* \tilde{d} \varphi_n - \chi_{n,1} \chi_{n,2} \tilde{d}^* \tilde{d} \psi_{\varepsilon_n, n, \xi_n} &= [-i\chi_{n,2} \partial_1 \chi_{n,1} + \chi_{n,1} \partial_2 \chi_{n,2}] \tilde{d}^* \psi_{\varepsilon_n, n, \xi_n} + \\ &\quad [-i\chi_{n,2} \partial_1 \chi_{n,1} - \chi_{n,1} \partial_2 \chi_{n,2}] \tilde{d} \psi_{\varepsilon_n, n, \xi_n} + \\ &\quad [-\chi_{n,2} \partial_1^2 \chi_{n,1} - \chi_{n,1} \partial_2^2 \chi_{n,2}] \psi_{\varepsilon_n, n, \xi_n}, \end{aligned}$$

with, using (47) and (48),

$$\begin{aligned} &\|[-i\chi_{n,2} \partial_1 \chi_{n,1} + \chi_{n,1} \partial_2 \chi_{n,2}] \tilde{d}^* \psi_{\varepsilon_n, n, \xi_n}\| \\ &\leq \sqrt{2(n+1)B_0} \|[-i\chi_{n,2} \partial_1 \chi_{n,1} + \chi_{n,1} \partial_2 \chi_{n,2}] \psi_{\varepsilon_n, n+1, \xi_n}\| \\ &\quad + \frac{\varepsilon_n}{2B_0} \|[-i\chi_{n,2} \partial_1 \chi_{n,1} + \chi_{n,1} \partial_2 \chi_{n,2}] \psi_{\varepsilon_n, n, \xi_n}\| \\ &\leq 2\sqrt{2(n+1)B_0} r_n^{-1} \|\chi'\|_\infty \sqrt{2r_n} \|\phi_{n+1}\| + 2\frac{\varepsilon_n}{2B_0} r_n^{-1} \|\chi'\|_\infty \sqrt{2r_n} \|\phi_n\| \\ &\leq 2\sqrt{2} \|\chi'\|_\infty \left(B_0 \sqrt{\frac{2n+2}{n^{1+\epsilon}}} + \frac{\varepsilon_n}{2} \sqrt{\frac{B_0}{n^{1+\epsilon}}} \right) \sqrt{r_n}, \end{aligned}$$

$$\begin{aligned} &\|[-i\chi_{n,2} \partial_1 \chi_{n,1} - \chi_{n,1} \partial_2 \chi_{n,2}] \tilde{d} \psi_{\varepsilon_n, n, \xi_n}\| \\ &\leq \sqrt{2nB_0} \|[-i\chi_{n,2} \partial_1 \chi_{n,1} - \chi_{n,1} \partial_2 \chi_{n,2}] \psi_{\varepsilon_n, n-1, \xi_n}\| \\ &\quad + \frac{\varepsilon_n}{2B_0} \|[-i\chi_{n,2} \partial_1 \chi_{n,1} - \chi_{n,1} \partial_2 \chi_{n,2}] \psi_{\varepsilon_n, n, \xi_n}\| \\ &\leq 2\sqrt{2} \|\chi'\|_\infty \left(B_0 \sqrt{\frac{2n}{n^{1+\epsilon}}} + \frac{\varepsilon_n}{2} \sqrt{\frac{B_0}{n^{1+\epsilon}}} \right) \sqrt{r_n} \end{aligned}$$

and

$$\|[-\chi_{n,2} \partial_1^2 \chi_{n,1} - \chi_{n,1} \partial_2^2 \chi_{n,2}] \psi_{\varepsilon_n, n, \xi_n}\| \leq 2\sqrt{2} \|\chi''\|_\infty r_n^{-2} \sqrt{r_n}.$$

Thus, in view of condition (21) and estimate (53), we get

$$(55) \quad \|\tilde{d}^* \tilde{d} \varphi_n - \chi_{n,1} \chi_{n,2} \tilde{d}^* \tilde{d} \psi_{\varepsilon_n, n, \xi_n}\| / \|\varphi_n\| \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

For estimating the remaining term on the r.h.s of (54), we expand $V_{\mathcal{R}_n}$ up to second order and obtain, by (50) and (51), that

$$|[V_{\mathcal{R}_n}(\mathbf{x}) - V_n - \mathcal{E}_n(x_1 - x_{1,n})]\varphi_n(\mathbf{x})| \leq \|\text{Hess}(V_{\mathcal{R}_n})\|_2(\boldsymbol{\eta}_{\mathbf{x}, \mathbf{x}_n})|\mathbf{x} - \mathbf{x}_n|^2|\varphi_n(\mathbf{x})|,$$

with $\boldsymbol{\eta}_{\mathbf{x}, \mathbf{x}_n} \in [\mathbf{x}, \mathbf{x}_n]$. Because \mathcal{R}_n are rotations, we have that $\|\text{Hess}(V_{\mathcal{R}_n})\|_2(\cdot) = \|\text{Hess}(V)\|_2(\mathcal{R}_n \cdot)$ for $n \in \mathbb{N}$. Since $\|\text{Hess}(V)\|_2$ varies with rate ν along $\text{Im}(\gamma)$ and since, by (20) and (49), $r_n/|\mathbf{x}_n|^\nu \rightarrow 0$ as $n \rightarrow \infty$, we find a constant $C_{11} > 0$ such that for $n \in \mathbb{N}$ large enough

$$\|\text{Hess}(V_{\mathcal{R}_n})\|_2(\boldsymbol{\eta}) \leq C_{11}\|\text{Hess}(V_{\mathcal{R}_n})\|_2(\mathbf{x}_n), \quad \boldsymbol{\eta} \in B_{2r_n}(\mathbf{x}_n)$$

holds. As a consequence,

$$\begin{aligned} \|U_{\mathcal{R}_n}e^{-ig}[V_{\mathcal{R}_n} - V_n - \mathcal{E}_n(x_1 - x_{1,n})]\varphi_n\| &\leq 4C_{11}r_n^2\|\text{Hess}(V)\|_2(\mathcal{R}_n\mathbf{x}_n)\|\varphi_n\| \\ &\leq 4C_{11}\|\text{Hess}(V)\|_2(\mathbf{y}_n)\left(\frac{|V_n|}{B_0}\right)^{1+\epsilon}\|\varphi_n\| \end{aligned}$$

for $n \in \mathbb{N}$ large enough. In view of (19), we conclude that $(U_{\mathcal{R}_n}e^{-ig}\varphi_n)_{n \in \mathbb{N}}$ is a Weyl sequence for 0. \square

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APPENDIX A. ESSENTIAL SELF-ADJOINTNESS OF THE PAULI OPERATOR

In this section we recapitulate an argument, originally given in [8], for proving the essential self-adjointness of the Pauli operator. As we will see, this argumentation works also for the relaxed regularity conditions on B and V of Proposition 1.

For the proof we first note that for $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{C})$ we can write

$$\begin{aligned} [(-i\nabla - \mathbf{A})^2 + B]\varphi &= \sum_{k,l=1}^2 (-i\partial_k - A_k)\overline{C_{k,l}}(-i\partial_l - A_l)\varphi, \\ [(-i\nabla - \mathbf{A})^2 - B]\varphi &= \sum_{k,l=1}^2 (-i\partial_k - A_k)C_{k,l}(-i\partial_l - A_l)\varphi, \end{aligned}$$

where $C_{k,l}$ denote the coefficients of the symmetric non-negativ definite matrix

$$C = \mathbb{1} - \sigma_2 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = C^*.$$

Furthermore, along the proof we use the notation $B_R := \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| \leq R\}$ and $S_R := \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| = R\}$.

Proof of Proposition 1. Since H is a diagonal matrix operator, it suffices to show that both operators on the diagonal,

$$Q_\pm := [(-i\nabla - \mathbf{A})^2 \pm B + V],$$

are essentially self-adjoint on $C_0^\infty(\mathbb{R}^2, \mathbb{C})$. Because Q_\pm are symmetric on $C_0^\infty(\mathbb{R}^2, \mathbb{C})$, we have to show $Q_+^*\varphi = \pm i\varphi$ implies $\varphi \equiv 0$ for $\varphi \in \mathcal{D}(Q_+^*)$, respectively $Q_-^*\varphi = \pm i\varphi$ implies $\varphi \equiv 0$ for $\varphi \in \mathcal{D}(Q_-^*)$. We only treat the case $Q_-^*\varphi = i\varphi$ since the others are completely analogous. Let $\varphi \in \mathcal{D}(Q_-^*)$ be such that $Q_-^*\varphi = i\varphi$, then

$$(56) \quad [(-i\nabla - \mathbf{A})^2 - B + V]\varphi = i\varphi$$

holds in distributional sense. Due to elliptic regularity theory (see e.g. [6], [7]), we obtain that $\varphi \in C^2(\mathbb{R}^2, \mathbb{C})$ and that (56) holds strongly. Applying integration by parts results in

$$\begin{aligned}
 (57) \quad & \int_{B_R} \left[\sum_{k,l=1}^2 (-i \partial_k - A_k) C_{k,l} (-i \partial_l - A_l) \varphi \right] \bar{\varphi} \, d^2x \\
 &= \int_{B_R} \left[\sum_{k,l=1}^2 (-i \partial_k - A_k) \varphi C_{k,l} \overline{(-i \partial_l - A_l) \varphi} \right] d^2x \\
 &\quad + i \int_{S_R} \left[\sum_{k,l=1}^2 \nu_k C_{k,l} (-i \partial_l - A_l) \varphi \right] \bar{\varphi} \, dS,
 \end{aligned}$$

with $R > 0$, where $\nu_k(\mathbf{x}) = x_k/|\mathbf{x}|$ for $k = 1, 2$. By taking the imaginary part of (57), we conclude with (56) that

$$\int_{B_R} |\varphi|^2 \, d^2x = \int_{S_R} \left[\sum_{k,l=1}^2 \nu_k C_{k,l} (-i \partial_l - A_l) \varphi \right] \bar{\varphi} \, dS$$

for any $R > 0$. The Cauchy-Schwarz inequality yields

$$\int_{B_R} |\varphi|^2 \, d^2x \leq \left(\int_{S_R} \sum_{k,l=1}^2 (-i \partial_k - A_k) \varphi C_{k,l} \overline{(-i \partial_l - A_l) \varphi} \, dS \right)^{1/2} \left(\int_{S_R} |\varphi|^2 \, dS \right)^{1/2}.$$

Hence, it suffices to show that

$$(58) \quad \int_{\mathbb{R}^2} \sum_{k,l=1}^2 \frac{(-i \partial_k - A_k) \varphi C_{k,l} \overline{(-i \partial_l - A_l) \varphi}}{|\mathbf{x}|^2 + 1} \, d^2x < \infty$$

since this implies that $(1, \infty) \ni r \mapsto r^{-1} \int_{B_r} |\varphi|^2 \, d\mathbf{x}$ is an L^1 -function, i.e. $\varphi \equiv 0$. For (58) we consider the function

$$f(R) := \int_{B_R} \sum_{k,l=1}^2 \frac{(-i \partial_k - A_k) \varphi C_{k,l} \overline{(-i \partial_l - A_l) \varphi}}{|\mathbf{x}|^2 + 1} \, d^2x,$$

with $R > 0$. Using Equation (56) and integration by parts, we obtain, with $\zeta(\mathbf{x}) = (|\mathbf{x}|^2 + 1)^{-1}$ and $M \geq c + |d|$, that

$$\begin{aligned}
 f(R) - M \|\varphi\|^2 &\leq f(R) + \int_{B_R} \zeta V |\varphi|^2 \, d^2x \\
 &= \int_{B_R} \zeta (Q_-^* \varphi) \bar{\varphi} \, d^2x - i \int_{B_R} \left[\sum_{k,l=1}^2 (\partial_l \zeta) C_{k,l} (-i \partial_k - A_k) \varphi \right] \bar{\varphi} \, d^2x \\
 &\quad + i \int_{S_R} \zeta \left[\sum_{k,l=1}^2 \nu_l C_{k,l} (-i \partial_k - A_k) \varphi \right] \bar{\varphi} \, dS.
 \end{aligned}$$

By the estimates

$$\begin{aligned}
& \left| \int_{B_R} \left[\sum_{k,l=1}^2 (\partial_l \zeta) C_{k,l} (-i \partial_k - A_k) \varphi \right] \bar{\varphi} \, d^2 x \right| \\
& \leq \int_{B_R} 2\zeta^{1/2} \left| \sum_{k,l=1}^2 \nu_l C_{k,l} (-i \partial_k - A_k) \varphi \right| |\varphi| \, d^2 x \\
& \leq 2 \int_{B_R} \left[\sum_{k,l=1}^2 \zeta (-i \partial_k - A_k) \varphi C_{k,l} \overline{(-i \partial_l - A_l) \varphi} \right]^{1/2} |\varphi| \, d^2 x \\
& \leq 2[f(R)]^{1/2} \|\varphi\| \leq \frac{1}{2} f(R) + 2\|\varphi\|^2
\end{aligned}$$

and

$$\begin{aligned}
& \left| \int_{S_R} \zeta \left[\sum_{k,l=1}^2 \nu_k C_{k,l} (-i \partial_l - A_l) \varphi \right] \bar{\varphi} \, dS \right| \\
& \leq \int_{S_R} \zeta \left[\sum_{k,l=1}^2 (-i \partial_k - A_k) \varphi C_{k,l} \overline{(-i \partial_l - A_l) \varphi} \right]^{1/2} |\varphi| \, dS \\
& \leq \left(\int_{S_R} \sum_{k,l=1}^2 \zeta (-i \partial_k - A_k) \varphi C_{k,l} \overline{(-i \partial_l - A_l) \varphi} \, dS \right)^{1/2} \left(\int_{S_R} |\varphi|^2 \, dS \right)^{1/2} \\
& = \left(f'(R) \int_{S_R} |\varphi|^2 \, dS \right)^{1/2},
\end{aligned}$$

we conclude that

$$f(R) \leq 2(3+c)\|\varphi\|^2 + 2 \left(f'(R) \int_{S_R} |\varphi|^2 \, dS \right)^{1/2}.$$

If $f(R) = 0$ for all $R > 0$, then clearly (58) holds. If $f(R_0) > 0$ for some $R_0 > 0$, then $f(R) > 0$ for all $R > R_0$ and $f'(R)/f^2(R) \in L^1((R_0, \infty))$, implying that

$$\left(\frac{f'(R)}{f^2(R)} \int_{S_R} |\varphi|^2 \, dS \right)^{1/2} \in L^1((R_0, \infty)).$$

Hence, there exists a sequence $(R_n)_{n \in \mathbb{N}} \subset (0, \infty)$ such that $R_n \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\left(f'(R_n) \int_{S_{R_n}} |\varphi|^2 \, dS \right)^{1/2} \leq \frac{1}{4} f(R_n).$$

Therefore, we have $f(R_n) \leq 4(3+c)\|\varphi\|^2$ for all $n \in \mathbb{N}$, which implies (58) since $f(R)$ is a monotonically increasing function. \square

APPENDIX B. REMARKS ON LOCALLY COMPACT OPERATORS

Lemma 4. *Let A be a locally compact, self-adjoint operator on $L^2(\mathbb{R}^n, \mathbb{C}^m)$ with $n, m \geq 1$. Assume there is a function $W \in L_{loc}^\infty(\mathbb{R}^n, [0, \infty))$, with $W(\mathbf{x}) \rightarrow \infty$ as $|\mathbf{x}| \rightarrow \infty$, such that*

$$\|A\varphi\| \geq \|W\varphi\| \quad \text{for } \varphi \in \mathcal{D}(A).$$

Then $\sigma_{\text{ess}}(A) = \emptyset$, i.e. A has only discrete spectrum.

Proof. Assume $\lambda \in \sigma_{\text{ess}}(A) \subset \mathbb{R}$. Then there is a normalized sequence $(\varphi_n)_{n \in \mathbb{N}} \subset \mathcal{D}(A)$ such that $\varphi_n \rightharpoonup 0$ and $\|(A - \lambda)\varphi_n\| \rightarrow 0$ as $n \rightarrow \infty$. Let $R > 0$ be a fixed constant. We have

$$\begin{aligned} \|\chi_R \varphi_n\| &= \|\chi_R(A - i - \lambda)^{-1}(A - i - \lambda)\varphi_n\| \\ &\leq \|\chi_R(A - i - \lambda)^{-1}\| \|(A - \lambda)\varphi_n\| + \|\chi_R(A - i - \lambda)^{-1}\varphi_n\|, \end{aligned}$$

using the notation $\chi_R := \chi_{B_R(0)}$. Since $\chi_R(A - i - \lambda)^{-1}$ is compact, this inequality implies that $\|\chi_R \varphi_n\| \rightarrow 0$ as $n \rightarrow \infty$. Let $R > 0$ be so large that $W(\mathbf{x}) \geq 5|\lambda| + 1$ if $|\mathbf{x}| \geq R$. Choosing $N \in \mathbb{N}$ large enough, we can estimate, for $n \geq N$,

$$\begin{aligned} \|(A - \lambda)\varphi_n\| &\geq \|W\varphi_n\| - |\lambda| \\ &\geq \|W(\mathbb{1} - \chi_R)\varphi_n\| - \|W\chi_R\|_\infty \|\chi_R \varphi_n\| - |\lambda| \\ &\geq (5|\lambda| + 1)\|(\mathbb{1} - \chi_R)\varphi_n\| - \|W\chi_R\|_\infty \|\chi_R \varphi_n\| - |\lambda| \\ &\geq (|\lambda| + 1/2) - \|W\chi_R\|_\infty \|\chi_R \varphi_n\|. \end{aligned}$$

Hence, $\|(A - \lambda)\varphi_n\| \not\rightarrow 0$ as $n \rightarrow \infty$, which is a contradiction. \square

APPENDIX C. INTEGRAL ESTIMATES

Lemma 5. *Let ϑ_n be the n -th Hermite polynomial. Then, for any $\epsilon > 0$, it holds that*

$$\begin{aligned} \frac{1}{2^n n!} \int_{\sqrt{n^{1+\epsilon}}}^{\infty} |\vartheta_n(x)|^2 e^{-x^2} dx &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \\ \frac{1}{2^n n!} \int_{-\infty}^{-\sqrt{n^{1+\epsilon}}} |\vartheta_n(x)|^2 e^{-x^2} dx &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Proof. We only treat the first case since the second claim can be deduced from the first one by a symmetry argument. Due to the identity

$$\vartheta_n(x) = (-1)^n \sum_{k_1+2k_2=n} \frac{n!}{k_1!k_2!} (-1)^{k_1+k_2} (2x)^{k_1}$$

for the n -th Hermite polynomial (see e.g. [1]), we obtain for $|x| \geq 1$ the estimate

$$|\vartheta_n(x)| \leq \sum_{k_1+2k_2=n} \frac{n!}{k_1!k_2!} (2|x|)^{k_1} \leq \frac{n+1}{2} 2^n n! |x|^n.$$

Thus, for $n \in \mathbb{N}$ large enough we have

$$\begin{aligned} \frac{1}{2^n n!} \int_{\sqrt{n^{1+\epsilon}}}^{\infty} |\vartheta_n(x)|^2 e^{-x^2} dx &\leq \frac{(n+1)^2}{4} 2^n n! \int_{\sqrt{n^{1+\epsilon}}}^{\infty} x^{2n} e^{-x^2} dx \\ &\leq \frac{(n+1)^2}{4} 2^n n^{1+\epsilon} n! \exp(-n^{1+\epsilon}/2) \int_{\sqrt{n^{1+\epsilon}}}^{\infty} e^{-x^2/2} dx, \end{aligned}$$

and the r.h.s. tends to 0 as $n \rightarrow \infty$ by Stirling's Formula. \square

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